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Updown generation of Beatty sequences

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1. INTRODUCTION

1.1. Using the notation of floor and roof (see section 1.9) and introducing real parameters θ and γ (with $0<\theta<1$) we consider the differences

$$p_{\theta,\gamma}(n)=\lceil (n+1)\theta+\gamma\rceil-\lceil n\theta+\gamma\rceil$$

$$q_{\theta,\gamma}(n)=\lfloor (n+1)\theta+\gamma\rfloor-\lfloor n\theta+\gamma\rfloor$$

for any $n\in\mathcal{Z}$. Keeping θ and γ fixed, we can consider both $p_{\theta,\gamma}$ and $q_{\theta,\gamma}$ as two-sided infinite sequences, the entries of which are all zeros and ones. These sequences have been named after the Canadian mathematician S. Beatty. Of the many papers devoted to Beatty sequences we just quote [11, 7, 1], and further references can be obtained from those papers.

1.2. Beatty sequences got a renewed interest by R. Penrose's discovery (around 1975) of a very remarkable class of non-periodic tilings of the plane, which became generally known through M. Gardner's article [8]. These tilings show the kind of infinite non-periodic repetition of arbitrarily large finite sub-configurations that we know from the Beatty sequences, and actually Beatty sequences (for the particular case that θ equals the golden ratio number $\tau=(-1+\sqrt{5})/2=0.618034$) turn up at various places in the analysis of Penrose patterns.

A further analogy lies in the operations of inflation and deflation, which formed a very essential key to Penrose's discovery. There are similar operations for Beatty sequences, at least for the case $\theta = \tau$ and for some similar cases (see [1]). From now on we shall use the term Beatty sequence exclusively for the case $\theta = \tau$.

1.3. Beatty sequences and Penrose patterns are examples of more general structures in spaces with arbitrary finite dimension. We refer to [4] for a general survey, and to [3] for descriptions of how the Beatty sequences fit into the picture. These generalized Penrose patterns are often called "quasicrystals" since the remarkable discovery in 1984 by Shechtman et al. (see [10]) of quasi-crystalline material with an unprecedented five-fold symmetry in the X-ray diffraction pattern. It is very remarkable that at the time of the discovery, mathematical models for such quasicrystals based on Penrose patterns, had just been made available (cf. [9]).

1.4. The Penrose patterns are particularly pretty since there is a satisfactory system of matching rules for the pieces (the thick and the thin arrowed rhombus). Whatever tiling we build according to those matching rules is automatically a Penrose pattern.

For most other quasicrystals, and in particular for the Beatty sequences, such matching rules do not exist.

For the Beatty sequences there is another type of property that characterizes them among the set of all possible zero-one sequences: they are the sequences with predecessors of all orders (see section 4.6), with a notion of predecessor that stems from the deflation operation (section 2.2). The corresponding property holds for the Penrose pieces too, but that is more or less a triviality since having just one predecessor already implies satisfying the matching conditions.

1.5. Penrose's operation of deflation transforms finite tilings into finite tilings with a larger number of pieces. Applying standard selection principles this led Penrose, in a non-constructive manner, to his infinite tilings. The procedure was described in detail in [6]. There is also a more constructive way, which seems to have been discovered first by J. Conway (rediscovered in 1981 by the author, and later by H. Lalvani). Thus far there seems to be no published account of it. In this paper the procedure is given the name *updown generation*. The method produces the infinite structures on the basis of infinite paths through a simple directed graph. It can be applied in all cases where we have a deflation operation.

The relation between the infinite paths and the infinite patterns is one to one, with exceptions for some of the so-called singular cases.

Updown generation provides a new kind of characterization both for the Beatty sequences and for the Penrose patterns: they are exactly all the structures we can get through updown generation.

1.6. We have two very different ways to describe a particular Beatty sequence: one by selecting a value for the parameter γ of section 1.1, and another one by selecting an infinite path through the oriented graph. Establishing the relation between the two ways is the main theme of the present paper. A survey of the details will be given in sections 7.5 and 7.6.

If we have any finite zero-one sequence the question whether it can ever be extended to a complete Beatty sequence, can be settled by means of a simple algorithm (section 8). And we can find out exactly what parts of the full sequence is completely forced by the finite part we started from. Sometimes very large parts of a Beatty sequence are forced already by just two entries.

In a subsequent paper a similar program will be carried out for the case of the Penrose patterns.

1.7. Instead of the ordinary real numbers, this paper will use a system of doubled reals (section 3). It facilitates the discussion at several places. One of the advantages is that we can use a single P instead of separate notations for the p and q of section 1.1.

1.8. As a by-product of the characterization by means of infinite paths we show (section 7.6) a peculiar number system for the reals, where instead of the base $\frac{1}{2}$ of the binary number system we have the base $-\tau$.

1.9. Notation. The floor $\lfloor x \rfloor$ of a real number x is the largest integer $\leq x$, and the roof $\lceil x \rceil$ is the smallest integer $\geq x$.

As usual, $\mathscr{Z}$ and $\mathscr{R}$ stand for the set of integers and the set of reals. And $\mathscr{D}$ is the doubled set of reals (section 3). For the notations γ_-, γ_+ we also refer to section 3.

We use τ for the golden ratio number: $\tau = (-1+\sqrt{5})/2 = 0.618034$. As a warning it should be stated here that many authors denote this number by τ^{-1} instead of τ.

S stands for the set of all reals of the form $k + h\tau$ with $k, h \in \mathscr{Z}$. The elements of S are called *singular* numbers.

For the definition of the Beatty sequence P_x (with index x in $\mathscr{D}$), see section 4.1.

See section 3.4 for the function ent, section 3.5 for the sets I, A, B, C, section 3.6 for the function φ, section 3.7 for ψ_ξ, ψ_η and ψ_ζ, section 3.8 for the sets U, V, W, section 6.1 for $S(s)$, section 6.4 for Ω, section 7.3 for Φ_θ, section 7.4 for $\Psi(x)$, section 7.5 for Σ and Π.

2. DEFLATION OF ZERO-ONE SEQUENCES

2.1. We start recalling some of the material treated in [1] (but it should be noted that much of the material of that paper is contained in older work by others; see the references in [1]).

We consider finite or infinite sequences of zeros and ones. Such a sequence can be considered as a mapping from a domain to the set $\{0, 1\}$, where the domain is a finite or infinite interval in the set $\mathscr{Z}$ of all integers. If the domain is the full $\mathscr{Z}$ we have a two-sided infinite sequence, if it runs from some integer m to ∞ we have a one-sided infinite sequence stretching to the right, if it runs from $-\infty$ to some integer m we have a one-sided infinite sequence stretching to the left.

We shall use the word *index* for the variable running over the domain of the sequence, and the term *segment* if we restrict a sequence to some sub-interval of the domain.

If f is a zero-one sequence, with domain D, and if k is any integer, then we can define the shifted sequence g, defined by $g(n+k)=f(n)$ for all n with $n \in D$. Denoting the shift operator by T_k, we can write $g=T_k f$, and for the domain of g we can write $T_k D$ or $D+k$. And we say that f and g are *shift-equivalent*.

The class of all sequences which are shift-equivalent to a given f is called the *sequence pattern* of f.

If f and g are shift-equivalent then it is clear from the domains which k is involved in $g=T_k f$, unless the sequences are two-sided infinite.

Given any sequence pattern we build a new sequence pattern that is to be called its *successor*. It is obtained by replacing every 0 of the original sequence by 1, and every 1 by the pair 10. For example, 1100111 has 101011101010 as its successor. Note that these are patterns, not sequences. The string 1100111 is the pattern of infinitely many sequences, which we get by taking any integer m and defining $f(m)=1$, $f(m+1)=1$, $f(m+2)=0, \ldots, f(m+6)=1$.

A sequence g is called a successor of f if the sequence pattern of g is the successor of the one of f, and the inverse relation is indicated by the word predecessor. In the case that the index place 0 belongs to the domain of f we indicate a particular successor. We shall say that g is the *main successor* of f if the digit at index place 0 in f gives rise in g to a digit 1 at index place 0, or to a pair 10 at index places 0 and 1.

2.2. It is not hard to describe these things by means of formulas. For simplicity we restrict ourselves to the two-sided infinite case.

Let f be a two-sided infinite zero-one sequence. In order to count the number of ones in any interval, we define the function $F_f: \mathcal{Z} \to \mathcal{Z}$ recursively by

$$(2.2.1) \quad F_f(0)=0, F_f(m+1)-F_f(m)=f(m)+1 \quad (m \in \mathcal{Z}).$$

Next we define the function $g: \mathcal{Z} \to \mathcal{Z}$ by saying that, for any $n \in \mathcal{Z}$, $g(n)$ is the number of integers m such that $n=F_f(m)$; this number $g(n)$ is 0 or 1. The function g is the main successor of f.

In analogy to the terminology for Penrose tilings, the process of passing from a sequence to a successor is usually called *deflation*.

2.3. The rule by which we get successors is what is often called a *rewriting rule*. It can be described by $1 \to 10$, $0 \to 1$.

In [1] a slightly more general rewriting rule was considered. The results were expressed there for the special case $1 \to 100$, $0 \to 10$. The changes that have to be made for the case $1 \to 10$, $0 \to 1$ are quite trivial.

The rewriting rule $0 \to 1$, $1 \to 10$ has a biological interpretation, related to the one of Fibonacci. There is a population of an animal species living along a line. There are two kinds: infants (zeros) and adults (ones). After a unit of time every infant has turned into an adult, and every adult has produced a baby on the right.

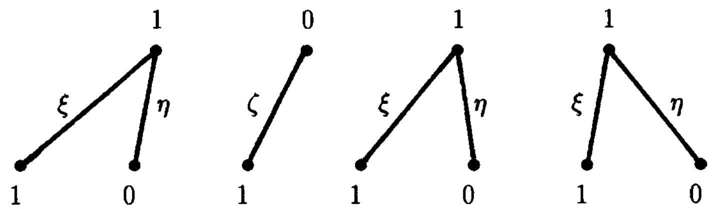

Fig. 1. An example of deflation, showing the connectors.

There seems to be a lack of symmetry in the fact that babies are born on the right and not on the left, and in that light the symmetries we have in the generated sequences may seem to be unexpected. It is not hard, however, to replace our production rule by a rule that preserves symmetry better. We replace every 1 by a pair of symbols Aa, and similarly every 0 by Bb. Now the deflation is described by the rewriting rules

$$A \to aB,\ a \to bA,\ B \to a,\ b \to A.$$

We shall not pursue this idea in this paper. We only remark that in [2], section 17, it was related to the study of stacks of hexagons in Penrose patterns.

2.4. If g is a successor of f then that fact can be described by a mapping $\mu : D_g \to D_f$, where D_g and D_f are the domains of g and f. For every $n \in D_g$ it indicates its "father" in D_f. This μ is weakly monotonic and such that for every $k \in D_f$ the number of $n \in D_g$ with $\mu(n) = k$ is either 1 or 2.

There are three kinds of pairs $(n, \mu(n))$. Those with $f(\mu(n)) = 1, g(n) = 1$ are called ξ-connections, those with $f(\mu(n)) = 1, g(n) = 0$ are called η-connections, and ζ-connections are those with $f(\mu(n)) = 0, g(n) = 1$. The symbols ξ, η, ζ are called *connectors.*

The example in figure 1 will speak for itself. The top row is D_f, the bottom row D_g.

3. THE DOUBLED SET OF REALS

3.1. We introduce the set $\mathscr{D}$ by duplication of the set of reals. We take two symbols *"left"* and *"right"*, and define $\mathscr{D}$ as the set of all pairs (p, *left*) and (p, *right*), where p runs through the reals. We shall also write $p-$ (or p_-) instead of (p, *left*) and $p+$ (or p_+) instead of (p, *right*).

In order to avoid confusion we shall use the letters $p, q, r, \ldots$, or greek letters, for reals, and $x, y, \ldots$ for elements of $\mathscr{D}$.

We define the projection π of $\mathscr{D}$ onto $\mathscr{R}$ by $\pi(p_-) = p$, $\pi(p_+) = p$ for all p in $\mathscr{R}$.

3.2. The order in $\mathscr{D}$ is the obvious one. For every real number p we agree that $p_- < p_+$, if q is a real number $> p$ then we agree that $p_+ < q_-$, and we take the inequality relation to be transitive, of course.

In $\mathscr{D}$ one can introduce a topology by defining the open sets. A subset O of

$\mathscr{D}$ is called open if for every p_- in O there is an $x<p_-$ such that all y with $x<y<p_-$ belong to O, and for every p_+ in O there is an $x>p_+$ such that all y with $p_+<y<x$ belong to O.

If $p<q$, one can consider the set of all x in $\mathscr{D}$ with $p_+\le x\le q_-$; it is compact in the topology of $\mathscr{D}$.

3.3. We do not try to define addition or multiplication in $\mathscr{D}$; giving a sense to things like p_-+q_+ would be awkward. But there is no harm in forming $x+p$ (for $x\in\mathscr{D}$, $p\in\mathscr{R}$) with the obvious interpretation: $(q_+)+p=(q+p)_+$, $(q_-)+p=(q+p)_-$. And if p is a real number >0, then the product of p with an element of $\mathscr{D}$ will be given by the rules $p(q_+)=(pq)_+$, $p(q_-)=(pq)_-$; if $p<0$ by $p(q_+)=(pq)_-$, $p(q_-)=(pq)_+$. If $p=0$ the product px will not be defined.

3.4. The integral part ent(x) of an element of $\mathscr{D}$ is the (uniquely defined) integer n with $n_+\le x\le(n+1)_-$.

We note that, if $p\in\mathscr{R}$, we have $\lfloor p\rfloor=\mathrm{ent}(p_+)$ for the floor of p, and $\lceil p\rceil=\mathrm{ent}(p_-)+1$ for its roof.

The mapping ent of $\mathscr{D}$ onto $\mathscr{Z}$ is continuous in the sense of the topology of section 3.2.

3.5. In section 4 we describe the deflation of Beatty sequences by means of sets I, A, B, C and functions φ, ψ_ξ, ψ_η, ψ_ζ. We shall display their definitions and some of their properties here.

We use the notation I, A, B, C for particular subsets (closed intervals) of $\mathscr{D}$:

$$I=\{x\in\mathscr{D}\,|\,(-\tau)_+\le x\le(1-\tau)_-\}$$

$$A=\{x\in\mathscr{D}\,|\,(-\tau)_+\le x\le(1-2\tau)_-\}$$

$$B=\{x\in\mathscr{D}\,|\,(1-2\tau)_+\le x\le 0_-\}$$

$$C=\{x\in\mathscr{D}\,|\,0_+\le x\le(1-\tau)_-\}.$$

Note that I is the disjoint union of A, B, and C.

3.6. The special function $\varphi: I\to I$ (see figure 2) is defined by

(3.6.1) $\varphi(x)=1-x/\tau+\mathrm{ent}((x/\tau)-\tau)$.

This function φ is continuous in the sense of the topology of section 3.2. We note that

(3.6.2) $\varphi(A)=A\cup B$, $\varphi(B)=C$, $\varphi(C)=A\cup B$.

We also note that for all $x\in I$ we have

(3.6.3) $-(x/\tau)=\varepsilon+\varphi(x)$,

with $\varepsilon=1$ if $x\in A$, $\varepsilon=0$ if $x\in B\cup C$.

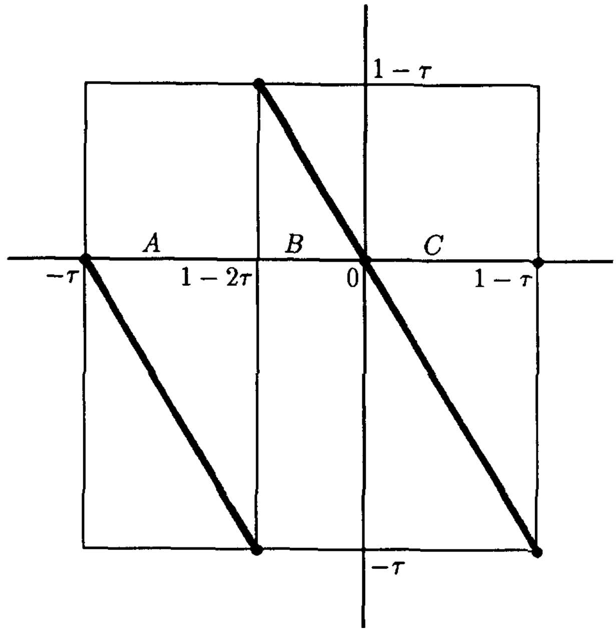

Fig. 2. Graph of φ.

3.7. We introduce auxilliary functions ψ_ξ, ψ_η, ψ_ζ by

$$\psi_\xi : A \cup B \to A, \ \psi_\xi(x) = -\tau x - \tau \quad (x \in A \cup B),$$

$$\psi_\eta : A \cup B \to C, \ \psi_\eta(x) = -\tau x \quad (x \in A \cup B),$$

$$\psi_\zeta : C \to B, \ \psi_\zeta(x) = -\tau x \quad (x \in C).$$

They form a kind of inverse to φ. We have

$$x = \psi_\xi(\varphi(x)) \quad (x \in A),$$

$$x = \psi_\eta(\varphi(x)) \quad (x \in C),$$

$$x = \psi_\zeta(\varphi(x)) \quad (x \in B).$$

Conversely

$$\varphi(\psi_\xi(y)) = y \quad (y \in A \cup B),$$

$$\varphi(\psi_\eta(y)) = y \quad (y \in A \cup B),$$

$$\varphi(\psi_\zeta(y)) = y \quad (y \in C).$$

If $x \in I$ and $\phi(x) = y$ then $y \in A \cup B$ or $y \in C$. In the first case $x = \psi_\xi(y)$ (if $x \in A$) or $x = \psi_\eta(y)$ (if $x \in B$), in the second case $x = \psi_\zeta(y)$.

3.8. We shall need some details about the action of the function φ on the *singular* elements of I, i.e. the elements of I whose projection π lies in S. The set of all singular elements in I can be parametrized by x_n and y_n, where n runs through the set of all integers:

$$x_n = (\lfloor (n-1)\tau \rfloor + 1 - n\tau)_-,\ y_n = (\lceil (n-1)\tau \rceil - n\tau)_+.$$

We have $\pi(x_n) = \pi(y_n)$ for all $n \neq 1$, and $x_1 = (1-\tau)_-$, $y_1 = (-\tau)_+$.

We split the set of singular elements into two disjoint parts. U is the set of all x_n and y_n with $n > 0$, and V is the set of all x_n and y_n with $n \geq 0$. The set of all non-singular elements of I will be denoted by W, so I is partitioned into U, V, W.

The action of φ on the singular elements is

$$\varphi(x_n) = y_{\lfloor (n-1)\tau \rfloor + 1},\ \varphi(y_n) = x_{\lceil (n-1)\tau \rceil}) \quad (n \in \mathscr{Z}).$$

So except for $n = 1$ we always have $\varphi(x_n) = y_m$, $\varphi(y_n) = x_m$, with $m = \lceil (n-1)\tau \rceil$.

Both U and V are mapped into themselves by φ. Iteration of φ on U ultimately leads to the two points $(-1+\tau)_-$ or $(-1+\tau)_+$, which are mapped onto each other. The last stages in the iteration are, when we omit the subscripts:

$$-3+5\tau \to -2+3\tau,\ -3+4\tau \to -2+3\tau,\ -2+3\tau \to -1+2\tau,$$

$$-1+2\tau \to -1+\tau,\ -1+\tau \to -1+\tau.$$

In V the iteration ultimately leads to the points 0_- and 0_+, which are again mapped onto each other. At the end of the iteration we can no longer afford to omit the subscripts: we have

$$(1-2\tau)_+ \to (1-\tau)_- \to (-\tau)_+,$$

$$(1-2\tau)_- \to (-\tau)_+,$$

$$(-\tau)_+ \to 0_- \to 0_+ \to 0_- \ldots$$

and all other elements of V ultimately lead to either $(1-2\tau)_-$ or $(1-2\tau)_+$, and from there to 0_- and 0_+.

4. BEATTY SEQUENCES AND THEIR DEFLATIONS

4.1. For any $x \in \mathscr{D}$ we define the two-sided infinite sequence P_x by

(4.1.1) $\quad P_x(n) = \mathrm{ent}((n+1)\tau + x) - \mathrm{ent}(n\tau + x).$

These sequences are called Beatty sequences. The definition captures both the p and the q of section 1.1. If $\gamma \in \mathscr{R}$ then

$$p_{\tau,\gamma} = P_{\gamma-},\ q_{\tau,\gamma} = P_{\gamma+}.$$

We note that P_x depends on x mod 1 only: if $k \in \mathscr{Z}$, $x \in \mathscr{D}$ then $P_{x+k} = P_x$. And we note that P_x and $P_{x+h\tau}$ are shift-equivalent (see section 2.1) for all $x \in \mathscr{D}$ and all integers h, since $P_{x+h\tau}(n) = P_x(n+h)$ for all integers n.

If $\gamma \in \mathscr{R}$ then $P_{\gamma-}$ and $P_{\gamma+}$ are different if and only if γ is singular (see section 3.8). If γ is singular then $P_{\gamma-}$ and $P_{\gamma+}$ differ only at two places. With $\gamma = k + h\tau$ we have

$$P_{\gamma-}(-1-h) = 0,\ P_{\gamma+}(-1-h) = 1,\ P_{\gamma-}(-h) = 1,\ P_{\gamma+}(-h) = 0.$$

As examples we display a part of P_{0-} and P_{0+} (the entries at index place 0 are printed in heavy type)

P_{0-}: ...01011010110110101101011011010110110101011010...

P_{0+}: ...01011010110110101101101011010110110101011010...

Symmetry can be studied by means of the formula $P_x(-n)=P_{-x}(n-1)$. P_{0-} and P_{0+} are not completely symmetric: there is a symmetry deficiency at index places -1 and 0. The only cases where P_x is properly symmetric are those where x has the form $u+k+h\tau$, where k and h are integers, and u has one of the three values $-1/2$, $\tau/2$, $(-1+\tau)/2$ (these cases are all non-singular, so we need not distinguish between $P_{\gamma-}$ and $P_{\gamma+}$). In particular

$P_{-1/2}$: ...101011011010110110101101011011010110110101...

$P_{\tau/2}$: ...0110110101101011011010110110101101011010110110...

$P_{(-1+\tau)/2}$: ...1101011011010110101101101011010110110101011...

The first one is symmetric around the point $-1/2$, the two others around -1.

4.2. Let $x\in\mathscr{D}$, $y\in\mathscr{D}$ and consider their projections $\pi(x)$ and $\pi(y)$ (see section 3.1). The answer to the question whether P_x and P_y are equal depends on whether $\pi(x)$ and $\pi(y)$ are in the singular set S or not (see section 1.9).

We have $P_x=P_y$ if and only if we are in one of the following cases:

(i) $\pi(x)-\pi(y)\in\mathscr{Z}$ and $\pi(x)\notin S$ (and therefore $\pi(y)\notin S$),

(ii) $x=\gamma_-$, $y=\delta_-$, where $\gamma\in S$, $\delta\in S$ and $\gamma-\delta\in\mathscr{Z}$,

(iii) $x=\gamma_+$, $y=\delta_+$, where $\gamma\in S$, $\delta\in S$ and $\gamma-\delta\in\mathscr{Z}$,

4.3 With the notation of section 4.2 it can be verified that P_x and P_y are shift-equivalent (section 2.1) if and only if we are in one of the following cases:

(i) $\pi(x)-\pi(y)\in S$, and $\pi(x)\notin S$ (whence $\pi(y)\notin S$),

(ii) $x=\gamma_-$, $y=\delta_-$, where $\gamma\in S$, $\delta\in S$, and $\gamma-\delta\in S$,

(iii) $x=\gamma_+$, $y=\delta_+$, where $\gamma\in S$, $\delta\in S$, and $\gamma-\delta\in S$.

4.4 The successors of Beatty sequences are again Beatty sequences. In particular we have for the main successor:

THEOREM. If $x\in\mathscr{D}$ then the main successor of P_x is P_y, where $y=-\tau x+$ $+\tau$ ent(x).

PROOF. We apply the result of section 2.2, where we specialize f to P_x. As the solution of (2.2.1) we find

$$F_f(m)=\mathrm{ent}(m\tau+x)+m-\mathrm{ent}(x)\quad(m\in\mathscr{Z}).$$

So the main successor of P_x is g, where $g(n)=1$ if there exists exactly one m with $\mathrm{ent}(m\tau+x)+m-\mathrm{ent}(x)=n$, and 0 otherwise. Putting $y=-\tau x+\tau\,\mathrm{ent}(x)$, $z=y+n\tau$, we reduce the condition $\mathrm{ent}(m\tau+x)+m-\mathrm{ent}(x)=n$ to $\mathrm{ent}((m-$ $-z)/\tau)=0$. So $g(n)=1$ if and only if z belongs to some interval $(m-\tau)_+\le$

$\leq z \leq m_-$ with $m \in \mathcal{Z}$. This condition is equivalent to $\text{ent}(z+\tau) - \text{ent}(z) = 1$, which means $P_y(n) = 1$. This proves $g = P_y$.

4.5. We now start with some P_y and ask whether it is a successor of another Beatty sequence. It will not always be a main successor: that can only happen if $P_x(0) = 1$.

Since $P_x = P_{x+k}$ for all $k \in \mathcal{Z}$ we can restrict ourselves to an interval of length 1, for which we take the interval I (see section 3.5, also for the partitioning of I into A, B and C).

We shall show that P_x is a particular successor (but not always the main successor) of $P_{\varphi(x)}$, where φ is the mapping of I into I introduced in section 3.6.

THEOREM. If $x \in A$ we have $x = \psi_\xi(\varphi(x))$, $P_x(0) = 1$, $P_{\varphi(x)}(0) = 1$, and P_x is the main successor of $P_{\varphi(x)}$.

If $x \in B$ we have $x = \psi_\zeta(\varphi(x))$, $P_x(0) = 1$, $P_{\varphi(x)}(0) = 0$, and P_x is the main successor of $P_{\varphi(x)}$.

If $x \in C$ we have $x = \psi_\eta(\varphi(x))$, $P_x(0) = 0$, $P_x(-1) = 1$, $P_{\varphi(x)}(0) = 1$, and $P_{x-\tau}$ is the main successor of $P_{\varphi(x)}$. This main successor is obtained from P_x by a shift: $P_{x-\tau}(n) = P_x(n-1)$ for all $n \in \mathcal{Z}$.

PROOF. The statements on the ψ's follow from section 3.7. The special values of the P's at 0 and -1 follow from (4.1.1).

By section 4.4, the main successor of $P_{\varphi(x)}$ is P_y, where $y = -\tau\varphi(x) + \tau\,\text{ent}(\varphi(x))$. We have $\text{ent}(\varphi(x)) = 1$ if $x \in B$, and $\text{ent}(\varphi(x)) = 0$ if $x \in A \cup C$. So from (3.6.3) we get $y = x$ if $x \in A \cup B$, and $y = x - \tau$ if $x \in C$.

4.6. We conclude that every P_x has predecessors of all orders, which means that P_x is a successor of a second two-sided infinite sequence, where this second sequence is a successor of a third one, the third is a successor of a fourth, etc.

It was the main result of [1] that the converse is also true: if a two-sided infinite sequence of zeros and ones has predecessors of all orders, then it is a Beatty sequence. In the present paper that result will be established independently, in a quite explicit form.

5. GENERATION BY MEANS OF CONNECTOR SEQUENCES

5.1. We shall describe a process, to be called *updown generation*. It can be applied to arbitrary rewriting rules (and, as we can learn from the Penrose patterns, the notion of rewriting rule can be taken in a wider sense than in language theory). Here it will be described for our rules $0 \to 1$, $1 \to 10$.

5.2. We first describe what can be called *topdown generation*. We start with a sequence pattern consisting of a single digit, then take its successor, the successor of the successor, etc. An example is shown in figure 3.

5.3. Let us focus our attention to an arbitrary entry of the bottom sequence, and trace the sequence of its ancestors: its father (see section 2.4), the

1

10

101

10110

10110101

1011010110110

101101011011010110101

Fig. 3. Topdown generation.

father of its father, etc. In figure 4 we have an example. The digit we selected in the bottom row is in heavy type. The predecessor sequences have been positioned in such a way that the sequence of ancestors lies in a column. To the right of that column we have written the connectors (see section 2.4), always halfway between father and child. Reading from bottom to top, we get the sequence of connectors $\eta\xi\xi\zeta\eta\xi$.

If we give the connector sequence, then we need not state whether the digit we started from was a 0 or a 1, since this information is provided by the first entry of the connector sequence. It is a 0 if and only if that first entry is η.

5.4. What we shall call *updown generation* is the way to get the bottom sequence from the sequence of connectors. The connector sequence determines the whole column, and then topdown generation determines the full pyramid.

We have to say how sequences are to be indexed. We do the obvious thing: in each row we always give the index 0 to the digit in heavy type. It follows that in the example of figure 4 the index in the bottom row runs from -9 to $+11$.

1
ξ
1 0
η
1 **0** 1
ζ
10 **1** 10
ξ
101 **1** 0101
ξ
10110 **1** 0110110
η
101101011 **0** 11010110101

Fig. 4. Updown generation by $\eta\xi\xi\zeta\eta\xi$.

The bottom row (with the indexing as just described) is what we call the zero-one sequence generated (in updown fashion) by the connector sequence. If the connector sequence is s, the zero-one sequence obtained by updown generation will be denoted by $S(s)$.

If s^* denotes the sequence we get from s by omitting its first entry, then the construction shows that $S(s)$ is a successor of $S(s^*)$. In the example of figure 4 we have $s^* = \xi\xi\zeta\eta\xi$, $S(s)$ is given by the bottom row, $S(s^*)$ by the row above it.

5.5. Connector sequences cannot be chosen arbitrarily: After a ξ or a η we can never have an η, and after a ζ we can only have an η. The possibilities can be indicated in a directed graph (also called finite automaton), see figure 5.

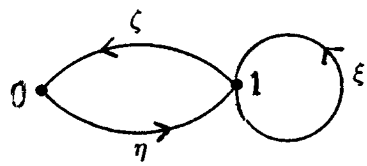

Fig. 5. The finite automaton belonging to the rewriting rule $1 \to 10$, $0 \to 1$.

The graph has two points, called 0 and 1. The arrows on the edges are just opposite to the direction in the rewriting rule. In the rewriting rule 1 leads to 1 by ξ and to 0 by η, and 0 leads to 1 by ζ. In the graph these orientations are reversed. This was done in order to enable us to say that a connector sequence can be used for updown generation if and only if it represents a *path* through the automaton.

A path like $\eta\xi\xi\zeta\eta\xi$ can also be described by the sequence of points 0111011 (the path starts at 0 and ends at 1). This is actually the central column of figure 4 from bottom to top.

5.6. If s and s' are two different paths through the automaton, then they generate two different sequences $S(s)$ and $S(s')$. In order to prove this, it suffices to show that the first entry of s is uniquely determined by $S(s)$. Reading $S(s)$ from right to left we uniquely split it into groups 10 and 1. This determines the row above it. And whether the connector at index place 0 is ξ, η or ζ, is according to whether the digit in heavy type in the bottom row (i.e. the one at index place zero) is a 1 from a group 10 (case ξ), a 0 from a group 10 (case η) or just a single 1 (case ζ).

5.7. If s is any finite connector sequence then the number of negative index places in the zero-one sequence $S(s)$ will be called $N(s)$, the number of positive index places $P(s)$. The example above shows $N(\eta\xi\xi\zeta\eta\xi) = 9$, $P(\eta\xi\xi\zeta\eta\xi) = 11$.

It is not hard to express $N(s)$ and $P(s)$ in terms of Fibonacci numbers. We denote these by a_n:

$$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13, \dots .$$

The Fibonacci numbers express the length of the sequences obtained by topdown generation. The k-th successor has length a_{k+2} if the top is 1, and a_{k+1} if the top is 0 (see the example of section 5.2).

This result on topdown generation can be used to write $N(s)$ and $P(s)$ as sums of Fibonacci numbers. The ξ's in the connector sequence contribute to $P(s)$, the η's to $N(s)$, the ζ's give no contribution at all. If the k-th entry of the connector sequence is a ξ then it contributes a_k to $P(s)$, if it is an η then it contributes a_{k+1} to $N(s)$. In the example $s=\eta\xi\xi\zeta\eta\xi$ we get $N(s)=a_2+a_6=9$, $P(s)=a_2+$ $+a_3+a_6=11$.

6. ZERO-ONE SEQUENCES GENERATED BY INFINITE CONNECTOR SEQUENCES

6.1. An infinite connector sequence represents an infinite path through the finite automaton of figure 5. Two kinds of infinite connector sequences will play a special role:

(i) sequences containing, from some point onwards, ξ's only,

(ii) sequences containing, from some point onwards, no ξ's at all, but just η's and ζ's alternatingly.

Infinite connector sequences of type (i) or (ii) will be called *singular*. The word singular will also be applied to the paths corresponding to them.

We shall describe how an infinite connector sequence always generates an infinite sequence of zeros and ones. Non-singular infinite connector sequences generate two-sided infinite sequences, but singular ones produce one-sided infinite sequences. In case (i) the sequence index will run from a non-positive number to $+\infty$, in case (ii) from $-\infty$ to some non-negative number. In particular, the connector sequence $\xi\xi\xi\xi\xi\xi\ldots$ generates a zero-one sequence running from 0 to $+\infty$, whereas both $\eta\zeta\eta\zeta\eta\zeta\ldots$ and $\zeta\eta\zeta\eta\zeta\eta\ldots$ generate zero-one sequences running from $-\infty$ to 0.

In an infinite connector sequence any finite initial segment is a finite connector sequence that determines a finite zero-one sequence by updown generation. We recall that the indices were placed in such a way that in the bottom row the starting point gets index 0. Because of this, it is obvious that if an initial segment is a part of a second initial segment, then the zero-one sequence generated (in updown style) by the first one is a segment of the zero-one sequence generated by the second one.

If we have an infinite connector sequence s and consider the successive initial segments $s_1, s_2, \ldots$, then for every k the zero-one sequence $S(s_k)$ is a segment of $S(s_{k+1})$. It follows that the sequence $S(s_1), S(s_2), S(s_3), \ldots$ has a limit (in the sense that the graph of that limit is the union of the graphs of the S_k's) The limit is what we shall call the zero-one sequence generated by the infinite connector sequence s. We denote it again by $S(s)$.

6.2. From the evaluation of $N(s_k)$ and $P(s_k)$ (see section 5.7) it follows that $S(s)$ has infinitely many negative index places if and only if s contains infinitely many η's, and infinitely many positive index places if and only if s contains infinitely many ξ's. So $S(s)$ is a two-sided infinite sequence if and only if s is non-singular; it is one-sided to the right in case (i) and to the left in case (ii).

If s is non-singular then the fact that for every k the sequence $S(s_k)$ is a successor of $S(s_k^*)$ (see section 5.4) reveals that the two-sided infinite sequence $S(s)$ has predecessors of all orders.

6.3 In order to illustrate the sequences generated by singular paths it suffices to take the special paths $\xi\xi\xi\xi\xi\xi\xi\ldots$, $\eta\zeta\eta\zeta\eta\zeta\eta\zeta\ldots$ and $\zeta\eta\zeta\eta\zeta\eta\zeta\eta\ldots$. Using heavy type for the digits we depart from, we find:

$$S(\xi\xi\xi\xi\xi\xi\xi\ldots)=\mathbf{1}011010110110101101 0\ldots,$$

$$S(\eta\zeta\eta\zeta\eta\zeta\eta\zeta\ldots)=\ldots 0101101011011010110\mathbf{110},$$

$$S(\zeta\eta\zeta\eta\zeta\eta\zeta\eta\ldots)=\ldots 01011010110110101101\mathbf{01}.$$

It follows from section 5.7 that these are the only paths for which the domain of $S(s)$ runs from 0 to ∞ or from $-\infty$ to 0.

The sequence generated by $\eta\zeta\eta\zeta\ldots$ can be continued to a two-sided infinite sequence with predecessors of all orders in only one way. It is clear that the part from 1 to ∞ has to have predecessors of all orders itself, so the only possibility is to take the sequence generated by $\xi\xi\xi\ldots$ and to shift it one place to the right.

For $\zeta\eta\zeta\eta\ldots$ we have the same thing.

If we start from $\xi\xi\xi\ldots$ we have two different possibilities for the continuation: one by means of $\eta\zeta\eta\zeta\ldots$ and one by means of $\zeta\eta\zeta\eta\ldots$ (both have to be shifted one place to the left).

$S(\eta\zeta\eta\zeta\ldots)$ is the restriction of P_{0+} (see section 4.1) to the non-positive index places, $S(\zeta\eta\zeta\eta\ldots)$ is the same restriction for P_{0-}, and $S(\xi\xi\xi\ldots)$ is the restriction of $P_{\tau+}$ to the non-negative index places (and we note that $P_{\tau+}$ can be obtained from P_{0+} by a shift). These facts can be shown by the argument of section 6.4; note that the theorem of section 4.5 shows that the predecessors of P_{0+} are $P_{0-}, P_{0+}, P_{0-}, \ldots$.

6.4. If p is a zero-one sequence with predecessors of all orders then p uniquely defines an infinite path $\Omega(p)$ in the automaton of figure 5; it describes the past history of the element at index place 0. We repeat the definition of Ω in a recursive form. Let q be the predecessor of p that is positioned in such a way that the entry $q(0)$ generates $p(0)$ (and possibly also $p(-1)$ or $p(1)$). If $q(0)=1, p(0)=1$, the path $\Omega(p)$ is ξ followed by the entire path $\Omega(q)$, if $q(0)=1$, $p(0)=0$, then $\Omega(p)$ is η followed by $\Omega(q)$, if $q(0)=0$, $p(0)=1$, then $\Omega(p)$ is ζ followed by $\Omega(q)$.

From what was said in sections 6.1 and 6.2 it follows that if p is a sequence with predecessors of all orders, then $S(\Omega(p))$ is either p itself or a one-sided infinite segment of p, running either from an index place ≤ 0 to ∞ or from $-\infty$ to some index place ≥ 0.

Conversely, if s is a non-singular path, then $S(s)$ is a two-sided infinite zero-one sequence with predecessors of all orders, and $s=\Omega(S(s))$. If s is singular of type (ii) then $S(s)$ can be completed uniquely to a sequence with predecessors of all orders that is mapped onto s by Ω, if s is singular of type (i) then there are exactly two such completions.

In particular we get: if p has predecessors of all orders and if $\Omega(p)$ is non-singular, then $S(\Omega(p))=p$.

7. GENERATED SEQUENCES AS PARTS OF BEATTY SEQUENCES

7.1. We again consider the example of section 5.4. Note that in the bottom row the digit in heavy type has index place 0. It is the digit the updown generation started from. The full bottom row, with index places -9 through $+11$, is the result of updown generation by means of the path $\eta\xi\xi\zeta\eta\xi$; that path can also be described by the sequences of graph vertices 0111011. We call that sequence of vertices a *path sequence.* The start and the finish of a path sequence θ are denoted by start(θ) and finish(θ), like in start(0111011)=0, finish(0111011=1. A path sequence can have length 1; in that case start and finish are equal. We now ask what is the set of all $x\in I$ (cf. section 3.5) with the property that the Beatty sequence P_x contains that bottom row at the same index places $-9,\dots,+11$. The set of those x will be denoted by $K(0111011)$. It will be clear how we attach such a set to every finite path in the automaton. As simplest cases we have $K(0)$ and $K(1)$, where the generated sequence is nothing but the digit we started from. Accordingly, $K(0)$ is the set of all $x\in I$ with $P_x(0)=0$, and $K(1)$ is the set of all $x\in I$ with $P_x(0)=1$. We know from section 4.5 that

(7.1.1) $K(0)=C,\ K(1)=A\cup B.$

7.2. Let $x\in K(0)$. It follows from sections 4.5 and 5.6 that P_x contains the bottom row in figure 4 if and only if P_y, where $y=\varphi(x)$ (and $x=\psi_\eta(y)$), contains the row above it (in both cases the word "contains" involves having the digit in heavy type at index place 0). And if y is any number in $K(1)$, then P_y contains that second row from the bottom if and only if P_z, with $z=\varphi(y)$ (and $y=\psi_\xi(z)$), contains the third row from the bottom. Combining these results we can evaluate the set of all $x\in I$ which are such that P_x contains the bottom row in figure 4. For that set, called $K(0111011)$ in section 7.1, we get

(7.2.1) $K(0111011)=\psi_\eta\psi_\xi\psi_\xi\psi_\zeta\psi_\eta\psi_\xi K(1).$

In this example it will be clear how this works. We have

(7.2.2) $K(0111011)=\psi_\eta(K(111011)),$

and the index η is the symbol attached to the path in the automaton of figure 5 leading from 0 (the first entry of the sequence 0111011 on the left) to 1 (the first entry of the sequence 111011 on the right). Finally the $K(1)$ on the right of (7.2.1) has as its argument the last entry of the path sequence 0111011; its value is given by (7.1.1).

Figure 6 displays the sets $K(\theta)$ for length 1 through 5, showing how extensions of the path lead to nested subdivisions of the interval. All intervals have to be taken as closed subintervals of I. The role of the arrows in figure 6 will be explained in section 7.3.

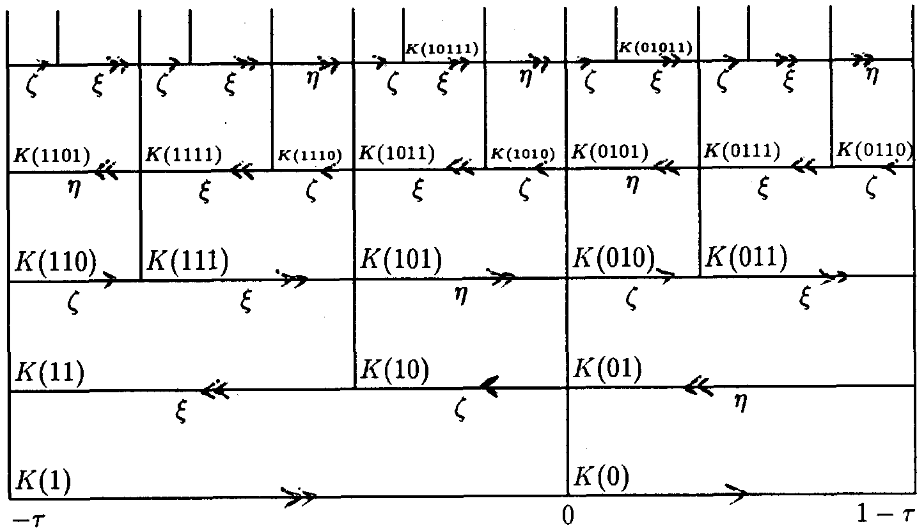

Fig. 6. The values of $K(\theta)$ for path sequences θ of length ≤ 5. The name of the interval in terms of path sequences is written above it; the greek letter printed under the interval is the name of the edge by which the path was extended.

The end-points of the intervals in figure 6 are the elements of V (section 3.8). Let us index the rows by values of j, using $j=0$ for the bottom row, $j=1$ for the one above it, etc. It is not hard to show that the j-th row contains as end-points all the x_n and y_n (cf. section 3.8) with $0 \leq n < a_{j+3}$, where the a_i are the Fibonacci numbers (section 5.7).

7.3. If θ is a path sequence of length $k>1$, the product of the ψ's belonging to the $k-1$ edges of the path (the factors in the product taken in the order of the path) will be denoted by Φ_θ. In our example we have $\theta = 0111011$, the path is $\eta\xi\xi\zeta\eta\xi$, and

$$\Phi_\theta = \Phi_{0111011} = \psi_\eta \psi_\xi \psi_\xi \psi_\zeta \psi_\eta \psi_\xi .$$

The generalization of (7.2.1) is obviously

(7.3.1) $K(\theta) = \Phi_\theta K(\mathrm{finish}(\theta))$,

and we note that $K(\theta)$ is just the range of Φ_θ. If θ is a path sequence with $\mathrm{finish}(\theta)=0$ we have

(7.3.2) $K(\theta) = K(\theta 1) = \Phi_\theta(C)$,

and if θ is a path sequence with $\mathrm{finish}(\theta)=1$ we obtain

(7.3.3) $K(\theta) = \Phi_\theta(A \cup B)$, $K(\theta 1) = \Phi_\theta A$, $K(\theta 0) = \Phi_\theta B$.

The function Φ_θ is linear with derivative $(-\tau)^{k-1}$, where k is the length of θ. In order to illustrate how $K(\theta)$ splits if the path is extended, we attach a single or double arrow to $K(\theta)$, either pointing to the left or to the right. The arrow is single if the endpoint finish(θ) is 0, and double if it is 1. The arrow points to the right if the length of θ is odd, and to the left if it is even. We shall use the term *dual deflation* for what happens to $K(\theta)$ when the path is extended. It is illustrated in figure 7.

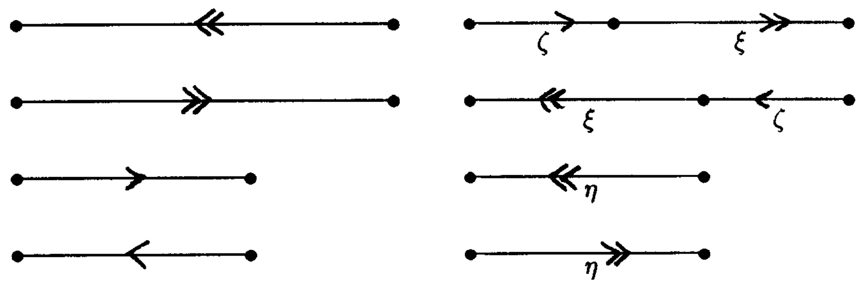

Fig. 7. The dual deflation subdivides the oriented closed intervals on the left as indicated on the right. The greek letters written under the intervals on the right indicate the edge by which the path is extended.

7.4. It was explained in section 6.1 how an infinite path through the graph of figure 5 determines, by updown generation, an infinite zero-one sequence $S(s)$, either one-sided or two-sided. Let θ be the sequence of vertices of such an infinite path, and for any positive integer k, let θ_k be the segment of the first k entries of θ. Obviously $K(\theta_1), K(\theta_2), K(\theta_3), \ldots$ is a nested sequence of closed intervals with length tending to 0. Therefore their intersection consist of at most two points of I: it is either a single point, some γ_- or γ_+, or a pair $\{\gamma_-, \gamma_+\}$.

The end-points of the intervals arising during the process of figure 6 are just all the points of the set V (see section 7.2). If and only if the nested sequence converges to one of these points we can have an intersection consisting of a single point. It happens if and only if the path is eventually alternating, i.e. having a tail $\eta\zeta\eta\zeta\eta\zeta\eta\zeta\ldots$. As the point of intersection we get a γ_+ if the ζ's occur at even index places (like in the path $\eta\zeta\eta\zeta\eta\zeta\eta\zeta\ldots$), and a γ_- if they occur at places with odd index.

In all other cases the intersection of the $K(\theta_k)$'s consists of a pair $\{\gamma_-, \gamma_+\}$. It is not hard to show that these points are elements of U if and only if the path is of the type (i) of section 6.1, e.g., ending with $\xi\xi\xi\xi\ldots$. The remaining cases are the non-singular paths (having infinitely many ξ's as well as infinitely many η's), and then the γ is non-singular.

The intersection of the $K(\theta_k)$'s is the set of all $x \in I$ which are such that P_x contains all what is generated by the finite initial segments of the path, i.e., the whole of $S(s)$ (section 6.1).

As remarked above, the nested sequence $K(\theta_1), K(\theta_2), K(\theta_3), \ldots$ does not always determine a unique element of I; sometimes there are two. But if we start from any $x \in I$ the sequence $\theta_1, \theta_2, \ldots$ with $x \in K(\theta_k)$ for all k is uniquely determined. These θ_k's are the initial segments of an infinite path sequence. This infinite path sequence determines an infinite path (see section 7.1 for the trivial relation between path sequence and path), and that path will be denoted by $\Psi(x)$.

7.5. We now summarize our results. The sets and mappings involved are shown in the diagram of figure 8.

$$\begin{tikzcd}
\pi(I) & I \arrow[l, "\pi"'] \arrow[r, "\Psi"] \arrow[d, "P"'] & \Pi \arrow[d, "S"] \\
 & \Sigma \arrow[ur, "\Omega"'] & S(\Pi)
\end{tikzcd}$$

Fig. 8. The sets and mappings involved.

To start with, I is the closed interval from $(-\tau)_+$ to $(1-\tau)_-$ in the doubled set of reals (section 3.5), and π is the projection on $\mathscr{R}$ (section 3.1), whence $\pi(I)$ is the closed interval from $-\tau$ to $1-\tau$ in the reals.

Next, P is the mapping that attaches to any $x \in I$ the Beatty sequence P_x. If x and y are different elements of I then $P_x = P_y$ only if $\pi(x)$ and $\pi(y)$ are equal to one and the same non-singular real (cf. section 4.2).

The letter Σ stands for the set of all two-sided infinite zero-one sequences with predecessors of all orders. From section 4.6 we know that P maps I into Σ.

The set of all infinite paths in the automaton of figure 5 is called Π. The mapping Ψ (section 7.4) attaches such a path to any $x \in I$. And Ω (section 6.4) attaches such a path to any element of Σ.

If $s \in \Pi$ then $S(s)$ is the one-sided or two-sided infinite sequence obtained by updown generation (section 6.1). The set of all $S(s)$ is called $S(\Pi)$.

I is partitioned into U, V and W (see section 3.8); W is the non-singular part.

We now formulate our main results.

THEOREM. (i) The mapping $S : \Pi \to S(\Pi)$ is bijective.

(ii) For all $p \in \Sigma$ the sequence $S(\Omega(p))$ is either p itself or the restriction of p to some one-sided infinite interval.

(iii) For all $x \in I$ we have $\Omega(P(x)) = \Psi(x)$.

(iv) $\Psi(I) = \Pi$.

(v) $P(I) = \Sigma$, i.e., every sequence with infinitely many predecessors is a Beatty sequence.

(vi) The partition $I = U \cup V \cup W$ leads to further partitions: $\pi(I)$ partitions into $\pi(U)$, $\pi(V)$, $\pi(W)$, next $\Psi(I)$ partitions into $\Psi(U)$, $\Psi(V)$, $\Psi(W)$, furthermore $P(I)$ partitions into $P(U)$, $P(V)$, $P(W)$, and $S(\Pi)$ into $S(\Psi(U))$, $S(\Psi(V))$, $S(\Psi(W))$.

(vii) $\Psi(U)$ is the set of all singular paths ending in $\xi\xi\xi\ldots$, $\Psi(V)$ is the set of all singular paths ending in $\eta\zeta\eta\zeta\ldots$, and $\Psi(W)$ is the set of all non-singular paths.

(viii) The relations between $\pi(W)$, $\Psi(W)$ and $P(W)$ are one-to-one. We have bijections $\alpha: \pi(W) \to \Psi(W)$, $\beta: \pi(W) \to P(W)$ such that on W we have $\alpha\pi = \Psi$, $\beta\pi = P$.

(ix) The restriction of Ω to $P(W)$ is a bijection $P(W) \to \Psi(W)$; its inverse is the restriction of S to $\Psi(W)$.

PROOF. (i) The argument was given in section 5.6.

(ii) See section 6.4.

(iii) Let $x \in I$, $s = \Omega(P_x)$. According to the definition of Ψ in section 7.4 we have to show that x lies in all $K(\theta_k)$'s defined by the finite initial segments of the path. Because of the definition of the $K(\theta_k)$'s (section 7.1), it suffices to check that $S(s)$ is contained in P_x. This follows from (ii).

(iv) It follows from the discussion in section 7.4 that every infinite path s leads to a nesting of closed intervals that have at least a point x of I in common. Obviously $\Psi(x) = s$.

(v) Take any $p \in \Sigma$, and put $s = \Omega(p)$. Since $s \in \Pi$ there is at least one $x \in I$ with $\Psi(x) = s$. By (i) we have $\Omega(P_x) = \Omega(p)$. If s is non-singular then $S(\Omega(P_x)) = = P_x$ and $S(\Omega(p)) = p$ by section 6.4, whence $p = P_x$. If s is singular we apply section 6.3. There we noted that there are at most two ways to complete $S(s)$ to an element of Σ. The cases that there are two ways are those where the path s ends in $\xi\xi\xi\ldots$, and then there is a real γ such that both for $x = \gamma_-$ and for $x = \gamma_+$ we have $x \in I$, $\Psi(x) = s$. These two possibilities for P_x form the two different extensions. So we have $p = P_x$ for one of the two.

(vi) First $\pi(U)$ is the set of γ with $-\tau \leq \gamma \leq 1 - \tau$, $\gamma = k + h\tau$, k, $h \in \mathcal{Z}$, $h > 0$; $\pi(V)$ similarly with $h \leq 0$, and $\pi(W)$ is the set of all other γ in that interval.

It follows from section 4.2 that $P(U)$, $P(V)$, $P(W)$ are pairwise disjoint. And the discussion of section 7.4 shows that $\Psi(U)$, $\Psi(V)$, $\Psi(W)$ are pairwise disjoint. The partition of $S(\Pi)$ follows from (i).

(vii) This was explained in section 7.4.

(viii) If $\gamma \in \pi(W)$ then every closed interval $K(\theta)$ contains either both γ_- and γ_+ or none of the two. This defines a path $\alpha(\gamma)$. And since a path sequence leads to a nested set of intervals with length converging to 0, different γ's give different $\alpha(\gamma)$'s.

The statements about β follow from section 4.2.

(ix) If $p \in P(W)$ then $\Omega(p) \in \Psi(W)$ by (vi), and by (v) $\Omega(p)$ is non-singular. By section 6.4 $S(\Omega(p)) = p$. And if $s \in \Psi(W)$ then s is non-singular by (vii), so $s = \Omega(S(s))$ by section 6.4.

7.6. Without putting them into the form of theorems and proofs, we mention some more facts that can be extracted from the previous sections.

(i) If $s \in \Psi(U)$ then $S(s)$ is a one-sided infinite sequence stretching to the right. There are exactly two elements of Σ that are continuations of $S(s)$. They are $P(\gamma_-)$ and $P(\gamma_+)$ for some $\gamma \in \pi(U)$, and γ_-, γ_+ are the only elements of I which are mapped by Ψ into s.

(ii) If $s \in \Psi(V)$ then $S(s)$ is a one-sided infinite sequence stretching to the left. There is exactly one element p of Σ that is a continuation of $S(s)$. It satisfies $\Omega(p) = s$.

(iii) There is a bijection $\alpha : \pi(U) \to \Psi(U)$ that satisfies $\alpha\pi = \Psi$ on U. And P maps U one-to-one onto $P(U)$.

(iv) The relations between V, $P(V)$ and $\Psi(V)$ by means of P, Ψ and Ω are one-to-one.

7.7. We have established in this paper that there are two different characterizations of the zero-one sequences with infinitely many predecessors. One is by the real parameter γ in the Beatty sequences, the other one by the infinite paths in the automaton of figure 5. And apart from some intricacies with the singular cases, the correspondence is one to one. This means that we have discovered a kind of new number system for the reals. It is similar to the binary number system, and has also exceptions where the representation is not unique. In the case of the binary number system the exceptional numbers have one finite and one infinite representation, but in this new system both representations of the exceptional numbers are finite. And the converse is true: whenever we have a finite development for a number, there is also a second one. The set of exceptional reals is the set E of all reals $\gamma \neq 0$ with the property that for some positive integer n we have $(-\tau)^n \in \pi(V)$ (it is easy to see that if this holds for $n = n_0$ then it remains true for all $n > n_0$).

THEOREM. Every real number γ can be written as the sum of an infinite series

$$(7.7.1) \quad \gamma = \sum_{j=m}^{\infty} \varepsilon_j (-\tau)^j$$

where m is some integer, and the coefficients $\varepsilon_m, \varepsilon_{m+1}, \ldots$ are all 0 or 1, with the extra condition that between any two 1's the number of 0's is even. If $\gamma \in E$ there are exactly two different possibilities for the coefficients, otherwise the representation (7.7.1) is unique.

PROOF. We may assume that $-\tau \leq \gamma \leq 1 - \tau$, for otherwise we can get into that interval by multiplication with a positive power of $-\tau$. Once γ lies in the interval we can get the ε's from (3.6.3), and that can be applied to γ_- and γ_+ separately. These can give rise to two different developments.

A satisfactory survey can be obtained from the relation between paths and interval nestings shown in figure 6. In the automaton of figure 5 every step ξ corresponds with a coefficient $\varepsilon = 1$, and the steps η, ζ correspond with $\varepsilon = 0$. So a path corresponds with a sequence of 0's and 1's with the property that between two 1's the number of 0's is even (but the path can start with and odd number of 0's).

Restricting (7.7.1) to $m = 1$ we can relate the partial sums of the series to figure 6. A partial sum is always the endpoint of an arrow: it is the tail in the case of a single arrow, the head in the case of a single one.

We now easily get to a full description of the developments (7.7.1) with $m=1$ and $-\tau\leq\gamma\leq 1-\tau$. We have $\Psi(\gamma_-)=\Psi(\gamma_+)$ if γ_- and γ_+ belong to $U\cup W$, and $\Psi(\gamma_-)\neq\Psi(\gamma_+)$ if γ_- and γ_+ are in V. The case $\gamma=0$ is different from the others: we have $\Psi(0_-)=\zeta\eta\zeta\eta\zeta\eta\ldots$, $\Psi(0_+)=\eta\zeta\eta\zeta\eta\zeta\ldots$, but this is the only case where two different paths lead to the same sequence of ε's. And $-\tau$, $1-\tau$ are exceptional since they are represented only once in the closed interval $(-\tau)_+\leq\gamma\leq(1-\tau)_-$. They are uniquely represented by (7.7.1) if we require $m=1$, but get a second representation if we allow $m<1$.

As a typical example we quote $\Psi((1-2\tau)_-)=\xi\xi\zeta\eta\zeta\eta\ldots$, $\Psi((1-2\tau)_+)==\zeta\eta\xi\zeta\eta\zeta\eta\ldots$, and these lead to the (finite) developments $1-2\tau=(-\tau)^1++(-\tau)^2=(-\tau)^3$.

7.8. In the case of the binary number system the rational numbers give rise to recurring developments. A sequence of ε's is called *recurring* if there are positive integers m and n such that $\varepsilon_i=\varepsilon_{i+n}$ for all $i>m$. In particular the finite developments are recurring.

It is not hard to show that the developments (7.7.1) are recurring if and only if γ has the form $(k+h\tau)/d$ with integers k, h, d $(d\neq 0)$.

Simple examples are $-1/2$, $\tau/2$, $(-1+\tau)/2$, which produce symmetric Beatty sequences (see section 4.1). Since

$$\varphi(-1/2)=(-1+\tau)/2,\ \varphi((-1+\tau)/2)=\tau/2,\ \varphi(\tau/2)=-1/2$$

these three are each other's deflation in cyclic order. The case $-1/2$ corresponds to the periodic path $\xi\zeta\eta\xi\zeta\eta\ldots$, and the sequence of ε's is 100100100.... For $\tau/2$ we get 010010010..., for $(-1+\tau)/2$ we get 001001001....

8. FORCINGS

8.1. If we prescribe a number of entries of a Beatty sequence then it can force a number of others, even infinitely many.

Given m different integers $n_1,\ldots,n_m$ and m values $\varepsilon_1,\ldots,\varepsilon_m$ (all 0 or 1), we can ask for the set $H(n_1,\varepsilon_1,\ldots,n_m,\varepsilon_m)$ of all $x\in I$ with

(8.1.1) $P_x(n_1)=\varepsilon_1,\ldots,P_x(n_m)=\varepsilon_m.$

We say that a pair (k,ε) is *forced* by (8.1.1) if $P_x(k)=\varepsilon$ for all x in that set. Obviously

(8.1.2) $H(n_1,\varepsilon_1,\ldots,n_m,\varepsilon_m)=H(n_1,\varepsilon_1)\cap\ldots\cap H(n_m,\varepsilon_m).$

8.2. For the description of forcings it is slightly simpler to consider the set I as a circle, pasting $(1-\tau)_-$ to $(-\tau)_+$. This is done by defining a new notion of interval $J(\alpha_+,\beta_-)$ for $\alpha_+,\beta_-\in I$, to be called a *circular interval*. If $\alpha<\beta$ it is just the set of all $x\in I$ with $\alpha_+\leq x\leq\beta_-$, if $\alpha>\beta$ it is the set of all $x\in I$ for which either $\alpha_+\leq x\leq(1-\tau)_-$ or $(-\tau)_+\leq x\leq\beta_-$; if $\alpha=\beta$ we just take $J(\alpha_+,\beta_-)=I$.

8.3. The sets $H(n, \varepsilon)$ are closed intervals of the type described in section 8.2. From (4.1.1) we can derive (with x_n and y_n as defined in section 3.8)

(8.3.1) $\quad H(n, 0) = J(y_n, x_{n+1}), \; H(n, 1) = J(y_{n+1}, x_n).$

From (8.1.2) we can now evaluate all other cases. We note that the intersections in (8.1.2) are either empty, or a single closed interval on the circle, or the union of two disjoint circular intervals. The circular intervals in (8.3.1) have length τ or $1-\tau$, and the intersection of a number of such intervals can never have more than two parts. We can even say that if not all of them have length τ then the intersection can have at most one part.

8.4. If the intersection is empty, then there are no Beatty sequences that satisfy (8.1.1). If the intersection is non-empty, there exist infinitely many Beatty sequences with (8.1.1).

If in (8.1.1) all ε's are 1, then it may happen, even with large values of m, that the intersection consists of two small intervals which are far apart. In that case (8.1.1) forces relatively few other pairs (k, ε), even though relatively few x satisfy (8.1.1). But in the more common case that the intersection consists of a single small circular interval (8.1.1) forces relatively many other pairs.

Let δ be the length of the single interval the intersection consists of, and assume $\delta < \tau$. Let M be a large positive number. Then, by the theory of equidistribution of the multiples mod 1 of an irrational number, any interval of M consecutive integers will contain about $2\delta M$ integers n for which neither $\pi(x_n)$ nor $\pi(x_{n+1})$ fall in that interval. If n has that property, the interval will be entirely covered by one of the two intervals in (8.3.1), and that means that either the pair $(n, 1)$ or the pair $(n, 0)$ has been forced.

8.5. Sometimes much can be forced by prescribing only two entries for a Beatty sequence. Let us take in (8.1.2) $m=2$, $n_1 = a$, $\varepsilon_1 = 0$, $n_2 = b$, $\varepsilon_2 = 1$. Now $J(y_a, x_{a+1})$ has length $1-\tau$, and $J(y_{b+1}, x_b)$ has length τ. The intersection will be small if the distance from $\pi(x_a)$ to $\pi(x_b)$ is small.

In particular each one of the intervals occurring in figure 6 can be forced by just two entries. As an example we start with the path $\zeta\eta\xi\zeta$. The set of all x for which P_x contains the updown generation is the sixth interval in the top row of figure 6. It is $J(y_2, x_{10})$, and can be seen as the intersection of $J(y_2, x_3)$ and $J(y_{11}, x_{10})$. This is what is forced by the condition $P_x(2) = 0$, $P_x(10) = 1$. The following lines shows part of what is forced:

110101101..10110101101..1011010110110101101..10110
10110110101101..10110101101..1011010110110101101..
10110101101..1011010110110101101..1011010110110101
101..10110101101..1011010110110101101..10110101101
1010110101101101011 01..1011010110110101101..101101
01101..1011010110110101101..1011010110110101101..1
0110101101..1011010110110101101..10110101101..1011
0101101101011 01..1011010110110101101..10110101101.

The index runs from -199 to $+200$, split up in groups of 50. The entries in heavy type are at index places 2 and 10; dots indicate index places j where $P_x(j)$ is not forced. The part from index place -2 to index place $+2$ is what is obtained by the updown generation of $\zeta\eta\xi\zeta$.

In general, forcing by means of just two conditions $P_x(n_1)=0$, $P_x(n_2)=1$ is particularly effective if n_2-n_1 equals a Fibonacci number a_k. The asymptotic density of the j for which the value of $P_x(j)$ is not forced equals $2\tau^k$.

8.6. The question for which x we have (8.1.1) can be visualized as follows. On the circle we take the points $x_{n_1},\dots,x_{n_m}$, and we attach the corresponding ε's to them. Now x satisfies (8.1.1) if and only if the arc running (in positive direction) from $\pi(x)$ to $\pi(x)+(1-\tau)$ contains all the 0's and its complement contains all the 1's (but this statement is inprecise in cases where a 0 or a 1 fall on the boundary of the arcs).

From this we can see at once which points are superfluous in the forcing. In particular we note that whatever is forced by m points is forced already by 4 points or fewer.

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